Michael Hanss

Applied Fuzzy Arithmetic

An Introduction with Engineering Applications

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So far as the laws of mathematics refer to reality, they are uncertain, and so far as they are certain, they do not refer to reality.

—Albert Einstein

Preface

In the second half of the past century, the theory of fuzzy sets arose as a new mathematical concept in the field of information processing, and it rapidly advanced to becoming a well-established scientific discipline and a challenging object of both theoretical research and practical application. Since its introduction by Lotfi A. Zadeh in 1965, enormous progress has been made and numerous subdomains of fuzzy set theory have emerged, such as fuzzy logic and approximate reasoning, fuzzy pattern recognition and fuzzy modeling, expert systems and fuzzy control, and fuzzy arithmetic. Compared to most other fields, fuzzy arithmetic has received little attention in recent years, and the scope of its practical application has barely exceeded the level of elementary academic examples. The reasons for this may be seen in the absence of a well-organized, systematic, and consistent elaboration of the theory of fuzzy arithmetic, the lack of practical approaches to its effective implementation, and the apparent underestimation of its potential for the solution of real-world problems.

The intention of this book is to fill this gap by providing a well-structured compendium that offers both a deeper knowledge about the theory of fuzzy arithmetic and an extensive view on its applications in the engineering sciences. The book is divided into two parts with chapter continuity. Part I, Chapters 1 to 5, gives an introduction to the theory of fuzzy arithmetic, which aims to present the subject in a well-organized and comprehensible form. The derivation of fuzzy arithmetic from the original fuzzy set theory and its evolution towards a successful implementation is presented with existing formulations of fuzzy arithmetic included and integrated in the overall context. Part II, Chapters 6 to 9, presents a diversified exposition of the application of fuzzy arithmetic, addressing different areas of the engineering sciences, such as mechanical, geotechnical, biomedical, and control engineering.

Chapter 1 gives a review of the fundamentals of fuzzy set theory by recalling the basic principles and definitions of classical set theory and introducing the fuzzy theoretical analogs as a generalization. In this connection, particular attention is given to the essentials of fuzzy set theory, while the often discussed associated areas of fuzzy logic and approximate reasoning are excluded for lack of relevancy. Under the heading of elementary fuzzy arithmetic, Chapter 2 introduces fuzzy numbers and fuzzy vectors as generalizations of their crisp counterparts, and presents different concepts for the realization of elementary binary operations of fuzzy arithmetic. Explicitly, the concepts of L-R fuzzy numbers, discretized fuzzy numbers, and decomposed fuzzy numbers are derived, thoroughly discussed and compared. With the objective of extending the applicability of fuzzy arithmetic from elementary binary operations to the evaluation of fuzzy rational expressions, standard fuzzy arithmetic is introduced in Chapter 3, where the attribute 'standard' characterizes the concept as the most commonly used formulation of fuzzy arithmetic. In addition to the definition and a case study of standard fuzzy arithmetic, this chapter deals with the exposure and discussion of the serious drawbacks and limitations, which distinctly challenge the practicality of this approach. To solve these limitations, Chapter 4 introduces the transformation method as the basis of an advanced fuzzy arithmetic which enables a significantly enhanced fuzzy arithmetical evaluation of arbitrary models with fuzzy-valued parameters. The chapter provides an exhaustive description of the different versions of the transformation method and concludes with an overview of efficient strategies for the implementation of the method. Forming a bridge to the applications section of this book, Chapter 5 places particular emphasis on the characteristic property of fuzzy arithmetic of being exceedingly well suited for the numerical solution of problems in consideration of uncertainty, providing an expedient classification of the uncertainty phenomena that can occur in engineering applications. Finally, amongst other additions to fuzzy arithmetic, the chapter focuses on a trend-setting approach to inverse fuzzy arithmetic, which is also based on the transformation method.

Marking the beginning of Part II, Chapter 6 presents a number of challenging applications of fuzzy arithmetic in the area of mechanical engineering. These range from the examination of structural joint connections with uncertain parameters, where the models are available in analytical form, to the simulation and analysis of the vibrations of an engine hood by the use of finite element software. Chapter 7 deals with applications in geotechnical engineering, focusing on problems of environmental importance such as flow processes of contaminant migration in porous media. The need to consider uncertainties in biomedical engineering is highlighted in Chapter 8, where the human glucose metabolism of patients with diabetes mellitus Type I takes a central position. Finally, the book is completed by an application of fuzzy arithmetic in the field of control engineering, which clearly differs from the well-established fuzzy-logic methods known as fuzzy control. This application consists of a fuzzy arithmetical approach to the linear quadratic regulator design for a system with uncertain model parameters.

In conclusion, it is a great pleasure for me to express my appreciation and thanks to a number of individuals who have helped me, either directly or indirectly, in the task of initiating and completing this book. I am grateful to all my colleagues, both past and present, at the Institut A für Mechanik, Universität Stuttgart, for the productive intellectual environment, the pleasant atmosphere and for their willingness to share their interests in mechanics and computational modeling, which lead to a notable number of research collaborations and documented results. In particular, I would like to thank Prof. Dr.-Ing. habil. Lothar Gaul, Director of the Institut A für Mechanik, Universität Stuttgart, for providing me the opportunity to realize my interests in fuzzy methods and to create a working environment that would enable the completion of this work. I record my sincere thanks to Prof. Dr.-Ing. Arnold Kistner, Institut A für Mechanik, Universität Stuttgart, for the many helpful suggestions to my research activities and especially for stimulating my interests for the theory of fuzzy sets a decade ago. I gratefully acknowledge the support of Prof. Dr. Michael Berthold, ALTANA-Lehrstuhl für Angewandte Informatik, Universität Konstanz and Past-President of the North American Fuzzy Information Processing Society for his kind willingness to review the manuscript. The debt I owe to Professor Patrick Selvadurai, William Scott Professor at the Department of Civil Engineering and Applied Mechanics, McGill University Montréal, Canada, is particularly substantial. Quite apart from his careful review of the manuscript and his inspiring scientific support, especially during his sojourns as an Alexander-von-Humboldt fellow and a Max-Planck awardee at the Institut A für Mechanik in Stuttgart, I am deeply grateful for the long lasting friendship of him and his family.

My special thanks and distinct appreciation goes to Mrs. Sally Selvadurai for her excellent and invaluable work of proof-reading the manuscript. Last but not least, I am indebted to my parents for the continued appreciation and support that I have received in all my educational pursuits.

Stuttgart, October 2004 *Michael Hanss*

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Part I

Introduction to Fuzzy Arithmetic

1

The Theory of Fuzzy Sets

1.1 Classical Sets

1.1.1 Terminology and Notation

According to the basic definitions of naïve set theory [19, 52], a *classical* or *crisp set* A can be defined as a collection of objects or elements x out of some universal set X, which are characterized by some well-defined common property. If an element shows this property, it belongs to the set A, and we can symbolically write $x \in A$. Otherwise, it is excluded, and we write $x \notin A$. Such a classical set, which is sometimes also referred to as an *ordinary set* [132], can be described in different ways. The first way, usually applied for sets with a finite, countable number of elements, is to explicitly list the elements that belong to the set, as in

$$A = \{11, 13, 17, 19\} . \tag{1.1}$$

In the second method, we define the set by giving the common property that the elements must possess in order to be included in the set. This condition for membership can be expressed by a statement $\mathcal{A}(x)$, which is true for a member element x, as in

$$\begin{aligned} A &= \{x \mid \mathcal{A}(x)\} \quad \text{with} \\ \mathcal{A}(x) &= \text{‘x is a prime number between ten and twenty’} . \end{aligned} \tag{1.2}$$

This formulation yields the same set A that was given by explicitly listing its elements in (1.1). In the third method, the member elements of the set can be defined by using a *characteristic function* μ_A, which as a mapping of the form

$$\mu_A : X \mapsto \{0, 1\} \tag{1.3}$$

indicates membership of the element $x \in X$ if $\mu_A(x) = 1$, and non-membership if $\mu_A(x) = 0$. For the set A of all prime numbers between ten and twenty, the characteristic function is

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad x = 11, 13, 17, 19 \\ 0 & \quad \text{otherwise} \quad . \end{cases} \tag{1.4}$$

1.1.2 Basic Definitions

In the following, some important terms of classical set theory are introduced and explained.

Universal Set

The *universal set* X is a nonempty set consisting of all possible elements x of relevance in a particular context. The characteristic function $\mu_X(x)$ of the universal set X is given by

$$\mu_X(x) = 1 \quad \forall\, x \in X \; . \tag{1.5}$$

If the universal set X is denumerable, i.e., countable and either finite or infinite, every subset A of X shall be called a *discrete set*, otherwise it shall be called a *continuous set.*

Empty Set

The *empty set* $\emptyset$ or $\{\}$ is a set that contains no elements. The characteristic function $\mu_\emptyset(x)$ of the empty set $\emptyset$ is given by

$$\mu_\emptyset(x) = 0 \quad \forall\, x \in X \; . \tag{1.6}$$

Cartesian Product

The n-fold *Cartesian product* $A_1 \times A_2 \times \ldots \times A_n$ of the sets $A_1, A_2, \ldots, A_n$, $n \in \mathbb{N}$, is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 \in A_1$, $x_2 \in A_2$, ..., $x_n \in A_n$. Symbolically, we can write this n-dimensional *product set* as

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in A_1 \;\wedge\; x_2 \in A_2 \;\wedge \ldots \wedge x_n \in A_n\} \; . \tag{1.7}$$

The Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ of the universal sets $X_1, X_2, \ldots, X_n$ is called *universal product set* or *universal product space*. If $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, ..., $A_n \subseteq X_n$, then

$$A_1 \times A_2 \times \ldots \times A_n \;\subseteq\; X_1 \times X_2 \times \ldots \times X_n \; . \tag{1.8}$$

If, for example, $n = 2$ and $X_1 = X_2 = \mathbb{R}$, the Cartesian product

$$X_1 \times X_2 = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1 \in \mathbb{R} \;\wedge\; x_2 \in \mathbb{R}\} \tag{1.9}$$

corresponds to the universal product set of all points on the Euclidean plane.

Classical Relations

The concept of a classical set, defined in its original sense for a one-dimensional universal set, can be generalized by the introduction of an n-dimensional set R, which is usually defined as the subset

$$R \subseteq A_1 \times A_2 \times \ldots \times A_n \tag{1.10}$$

of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of some (one-dimensional) sets $A_1, A_2, \ldots, A_n$. The set R is then called an *n-ary relation*, for it correlates the elements x_i of the single sets A_i, $i = 1, 2, \ldots, n$, in terms of its elements $(x_1, x_2, \ldots, x_n)$.

Without loss of generality, we can consider the sets $A_1, A_2, \ldots, A_n$ to be subsets of some universal sets $X_1, X_2, \ldots, X_n$, i.e., $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, ..., $A_n \subseteq X_n$, and from (1.10) and (1.8) follows

$$R \subseteq A_1 \times A_2 \times \ldots \times A_n \subseteq X_1 \times X_2 \times \ldots \times X_n \ . \tag{1.11}$$

Consequently, the set R can also be regarded as a relation that is defined in the universal product set $X_1 \times X_2 \times \ldots \times X_n$. In this definition of a relation, every regular, one-dimensional set is included as the special case of a unary relation with $n = 1$.

Following the definition of a classical, one-dimensional set in (1.2), an n-ary relation R can be defined by formulating the common property of the member elements $(x_1, x_2, \ldots, x_n)$, which corresponds to the relational condition $\mathcal{R}(x_1, x_2, \ldots, x_n)$ that has to be fulfilled by an element in order to be included in the set. We can write

$$R = \{(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n \mid \mathcal{R}(x_1, x_2, \ldots, x_n)\} \ . \tag{1.12}$$

According to the one-dimensional set, the member elements of the relation R can be defined by using the characteristic function μ_R, which as a mapping of the form

$$\mu_R : X_1 \times X_2 \times \ldots \times X_n \mapsto \{0, 1\} \tag{1.13}$$

indicates membership of the element $(x_1, x_2, \ldots, x_n)$ if $\mu_R(x_1, x_2, \ldots, x_n) = 1$, and non-membership if $\mu_R(x_1, x_2, \ldots, x_n) = 0$.

Example 1.1. As an example of a discrete relation, let us consider the binary relation

$$R_1 = \{(x_1, x_2) \in A_1 \times A_2 \mid x_1 > x_2\} \tag{1.14}$$

with $A_1 = \{6, 15, 30\} \subset \mathbb{N}$ and $A_2 = \{1, 2, 5, 10\} \subset \mathbb{N}$. Those pairs (x_1, x_2) of the product set $A_1 \times A_2$ which fulfill the relational condition

$$\mathcal{R}_1(x_1, x_2) = \text{`}x_1 \text{ is greater than } x_2\text{'} \tag{1.15}$$

belong to the relation, the others are excluded. The relation R_1 can be expressed in tabular form, as in Table 1.1, with the values of the characteristic function $\mu_{R_1}(x_1, x_2)$ as entries.

Example 1.2. As an example of a continuous relation, we consider the ternary relation

$$R_2 = \{(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3 \mid x_3 = x_1 + x_2\} \tag{1.16}$$

with $X_1 \times X_2 \times X_3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$. The member elements $(x_1, x_2, x_3) \in \mathbb{R}^3$ that fulfill the relational condition

$$\mathcal{R}_1(x_1, x_2, x_3) = \text{`}x_3 \text{ is equal to the sum of } x_1 \text{ and } x_2\text{'} \tag{1.17}$$

can be geometrically interpreted as the subset of points (x_1, x_2, x_3) in the Euclidean space that lie on the planar surface defined by the equation

$$x_1 + x_2 - x_3 = 0 \; . \tag{1.18}$$

Table 1.1. Discrete binary relation R_1 in tabular form.

R_1:

x_1 \ x_2	1	2	5	10
6	1	1	1	0
15	1	1	1	1
30	1	1	1	1
	$\mu_{R_1}(x_1, x_2)$			

Function

The *function* F is a set of ordered n-tuples $(x_1, x_2, \ldots, x_{n-1}, y) \in X_1 \times X_2 \times \ldots \times X_{n-1} \times Y$ such that for each $(x_1, x_2, \ldots, x_{n-1}) \in X_1 \times X_2 \times \ldots \times X_{n-1}$ there is a unique element $y \in Y$. Thus, the function F can be considered as a special case of an n-ary relation, where the member elements $(x_1, x_2, \ldots, x_{n-1}, y)$ are related by the functional dependence

$$y = F(x_1, x_2, \ldots, x_{n-1}) \; , \tag{1.19}$$

and F is a unique mapping of the form

$$F : X_1 \times X_2 \times \ldots \times X_{n-1} \mapsto Y \; . \tag{1.20}$$

Explicitly, the uniqueness condition can be formulated as follows:

$$(x_1, x_2, \ldots, x_{n-1}, y) \in F \;\wedge\; (x_1, x_2, \ldots, x_{n-1}, z) \in F \quad \Rightarrow \quad y = z \; . \tag{1.21}$$

The element y is called the *value* that the function F takes on at the *argument* $(x_1, x_2, \ldots, x_{n-1})$.

Power Set

The *power set* $\mathcal{P}(A)$ of a set A is the set of all possible subsets T of A. We can write

$$\mathcal{P}(A) = \{T \mid T \subseteq A\} \ . \tag{1.22}$$

In accordance with this formulation, the power set can be defined for an n-ary relation R, so $\mathcal{P}(R)$ is the set of all possible relations being subsets of R.

Cardinality of Classical Sets and Relations

For a discrete and finite set A, $A \subseteq X$, the *(absolute) cardinality* $\mathrm{card}(A) = |A|$ is defined as the number of elements of A. In terms of the characteristic function $\mu_A(x)$, $x \in X$, for the set A, the (absolute) cardinality can be formulated as

$$\mathrm{card}(A) = |A| = \sum_{x \in X} \mu_A(x) \ . \tag{1.23}$$

The *relative cardinality* $\mathrm{card}_X(A)$ of the set A with respect to a finite universal set X is defined as

$$\mathrm{card}_X(A) = \frac{\mathrm{card}(A)}{\mathrm{card}(X)} = \frac{\sum\limits_{x \in X} \mu_A(x)}{\sum\limits_{x \in X} 1} \ . \tag{1.24}$$

Similarly, for a continuous, finite set A, $A \subseteq X$, the (absolute) cardinality can be defined as

$$\mathrm{card}(A) = |A| = \int\limits_{x \in X} \mu_A(x) \, \mathrm{d}x \ , \tag{1.25}$$

and the relative cardinality as

$$\mathrm{card}_X(A) = \frac{\mathrm{card}(A)}{\mathrm{card}(X)} = \frac{\int\limits_{x \in X} \mu_A(x) \, \mathrm{d}x}{\int\limits_{x \in X} \mathrm{d}x} \ . \tag{1.26}$$

Obviously, the relative cardinalities $\mathrm{card}_X(X)$ and $\mathrm{card}_X(\emptyset)$ of the universal set X and the empty set $\emptyset$ are given by

$$\mathrm{card}_X(X) = 1 \qquad \text{and} \qquad \mathrm{card}_X(\emptyset) = 0 \ . \tag{1.27}$$

Generalizing the definition of the cardinality of sets by defining the cardinality of relations, we can formulate the absolute cardinality of a discrete n-ary relation R as

$$\mathrm{card}(R) = |R| = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \cdots \sum_{x_n \in X_n} \mu_R(x_1, x_2, \ldots, x_n) \ , \tag{1.28}$$

whereas for a continuous n-ary relation R, we get

$$\mathrm{card}(R) = |R| = \int\limits_{x_1 \in X_1} \int\limits_{x_2 \in X_2} \dots \int\limits_{x_n \in X_n} \mu_R(x_1, x_2, \dots, x_n)\, \mathrm{d}x_n\, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \ . \tag{1.29}$$

The corresponding relative cardinalities are defined in accordance with (1.24) and (1.26).

Convexity of Classical Sets and Relations

A continuous n-ary relation $R \subseteq \mathbb{R}^n$ is called *convex* if for every element $\boldsymbol{u} = (u_1, u_2, \dots, u_n) \in R$ and $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in R$

$$\lambda \boldsymbol{u} + (1 - \lambda)\boldsymbol{v} \in R \quad \forall\, \lambda \in [0, 1] \ . \tag{1.30}$$

From a geometrical point of view, a continuous set R of points in $\mathbb{R}^n$ is defined as convex if for every two points $\boldsymbol{u}, \boldsymbol{v} \in R$ the points on the connecting line between $\boldsymbol{u}$ and $\boldsymbol{v}$ also belong to R.

Example 1.3. Let us consider the continuous binary relation R given by the ellipsoidal set of points $(x_1, x_2) \in \mathbb{R}^2$ as shown in Fig. 1.1a. Obviously, the relation R is convex, since every point on the connecting line of two arbitrary points $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ of the relation R represents an element of the relation R. This, however, changes if we consider the binary relation S given by a formerly ellipsoidal set of points featuring an indentation as shown in Fig. 1.1b. This relation is not convex, since at least one pair of points $\boldsymbol{u}$ and $\boldsymbol{v}$ can be found such that parts of the connecting line of $\boldsymbol{u}$ and $\boldsymbol{v}$ do not belong to S.

The definition of convexity of relations includes, of course, the definition of convexity of regular, one-dimensional sets as the special case of unary relations.

Example 1.4. Let us consider the set

$$A = \{x \in \mathbb{R} \mid x \in [a, b], a < b\} \tag{1.31}$$

of all points $x \in \mathbb{R}$ within the closed interval $[a, b]$, $a < b$, as shown in Fig. 1.2a. Obviously, the set A is convex, since every point between two arbitrary points u and v of the set A represents an element of the interval $[a, b]$. If we consider, instead, the set

$$B = \{x \in \mathbb{R} \mid x \in [a, b] \ \wedge \ x \in [c, d], \ a < b < c < d\} \tag{1.32}$$

as shown in Fig. 1.2b, we can choose $u \in [a, b]$ and $v \in [c, d]$ to see that those points x between u and v with $x \in \,]b, c[$ do not belong to B. Thus, the set B is not convex.

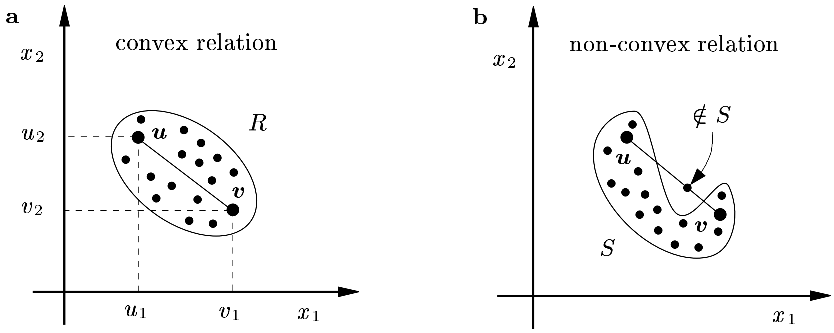

Fig. 1.1. Example of (**a**) a convex relation R and (**b**) a non-convex relation S in $\mathbb{R}^2$.

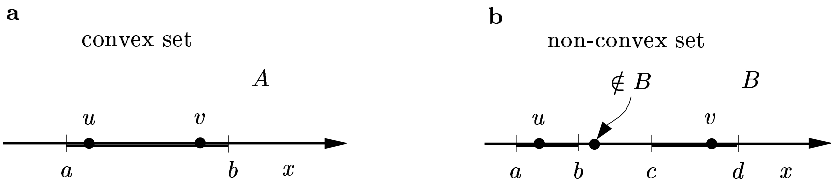

Fig. 1.2. Example of (**a**) a convex set A and (**b**) a non-convex set B in $\mathbb{R}$.

1.1.3 Operations for Domain-Compatible Classical Sets and Relations

Considering the definition of operations for classical sets and relations, we have to distinguish between operations for sets or relations that are domain-compatible and those that are not. Domain-compatible sets or relations are characterized by being defined on the same universal set or product set. Sets or relations that are not domain-compatible are defined on different universal sets or product sets. In the ensuing exposition, the most important operations for domain-compatible sets and relations are listed.

Inclusion (Containment)

A set A is included (contained) in or is equal to another set B if every element of A is also an element of B. With X being the universal set, we can symbolically write

$$A \subseteq B \quad \Leftrightarrow \quad \forall x \in X \; [x \in A \;\Rightarrow\; x \in B] \,. \tag{1.33}$$

If A is included in B, then A can be referred to as a *subset* of B, $A \subseteq B$, and B as a *superset* of A, $B \supseteq A$. If A is included in B and A is not equal to B, then A is said to be a *proper subset* of B. Symbolically, we can write

$$A \subset B \quad \Leftrightarrow \quad A \subseteq B \ \wedge \ A \neq B \ . \tag{1.34}$$

Recasting the definition of inclusion in terms of the characteristic functions $\mu_A(x)$ and $\mu_B(x)$ of the sets A and B, we can write

$$A \subseteq B \quad \Leftrightarrow \quad \mu_A(x) \leq \mu_B(x) \quad \forall\, x \in X \ . \tag{1.35}$$

In accordance with this formulation, we can give the following definition for the inclusion of two n-ary relations R and S, $R, S \subseteq X_1 \times X_2 \times \ldots \times X_n$:

$$R \subseteq S \quad \Leftrightarrow \quad \mu_R(x_1, x_2, \ldots, x_n) \leq \mu_S(x_1, x_2, \ldots, x_n) \qquad \forall\, (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n \ . \tag{1.36}$$

Equality

Two sets A and B, $A, B \subseteq X$, are equal if they contain exactly the same elements. Symbolically, we can write

$$A = B \quad \Leftrightarrow \quad \forall x \in X \ [x \in A \ \Leftrightarrow \ x \in B] \tag{1.37}$$

or, by using the definition of inclusion,

$$A = B \quad \Leftrightarrow \quad A \subseteq B \ \wedge \ B \subseteq A \ . \tag{1.38}$$

In terms of the characteristic functions $\mu_A(x)$ and $\mu_B(x)$ of the sets A and B, the definition can be rewritten as

$$A = B \quad \Leftrightarrow \quad \mu_A(x) = \mu_B(x) \quad \forall\, x \in X \ . \tag{1.39}$$

Accordingly, we can give the following definition for the equality of two n-ary relations R and S, $R, S \subseteq X_1 \times X_2 \times \ldots \times X_n$

$$R = S \quad \Leftrightarrow \quad \mu_R(x_1, x_2, \ldots, x_n) = \mu_S(x_1, x_2, \ldots, x_n) \qquad \forall\, (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n \ . \tag{1.40}$$

Complementation

The *complement* A^{c} of a set A is the set of all elements of the universal set X that are not members of A. Symbolically, we can write

$$A^{\mathrm{c}} = \{x \mid x \in X \ \wedge \ x \notin A\} \ . \tag{1.41}$$

In terms of the characteristic function $\mu_A(x)$ of the set A, the characteristic function $\mu_{A^{\mathrm{c}}}(x)$ of the complement A^{c} of A is defined as

$$\mu_{A^{\mathrm{c}}}(x) = \begin{cases} 1 & \text{if} \quad \mu_A(x) = 0 \\ 0 & \text{if} \quad \mu_A(x) = 1 \ . \end{cases} \tag{1.42}$$

The generalization of (1.42) to the definition of the complement R^{c} of an n-ary relation $R \subseteq X_1 \times X_2 \times \ldots \times X_n$ yields

$$\mu_{R^{\mathrm{c}}}(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if} \quad \mu_R(x_1, x_2, \ldots, x_n) = 0 \\ 0 & \text{if} \quad \mu_R(x_1, x_2, \ldots, x_n) = 1 \ . \end{cases} \tag{1.43}$$

Intersection

The *intersection* of two sets A and B, $A, B \subseteq X$, is a set $A \cap B$ that contains every element that is simultaneously a member of both the set A and the set B. Symbolically, we can write

$$A \cap B = \{x \in X \mid x \in A \ \wedge \ x \in B\} \ . \tag{1.44}$$

If the sets A and B are available in terms of the characteristic functions $\mu_A(x)$ and $\mu_B(x)$, the characteristic function $\mu_{A \cap B}(x)$ of the intersection $A \cap B$ is defined as

$$\mu_{A \cap B}(x) = \begin{cases} 1 & \text{if} \quad \mu_A(x) = 1 \ \wedge \ \mu_B(x) = 1 \\ 0 & \qquad \text{otherwise} \quad . \end{cases} \tag{1.45}$$

Generalizing (1.45) for the definition of the intersection $R \cap S$ of two n-ary relations R and S, $R, S \subseteq X_1 \times X_2 \times \ldots \times X_n$, we get

$$\mu_{R \cap S}(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if} \quad \mu_R(x_1, x_2, \ldots, x_n) = 1 \\ & \wedge \quad \mu_S(x_1, x_2, \ldots, x_n) = 1 \\ 0 & \qquad \text{otherwise} \quad . \end{cases} \tag{1.46}$$

Union

The *union* of two sets A and B, $A, B \subseteq X$, is a set $A \cup B$ that contains all the elements of either set A or set B. Symbolically, we can write

$$A \cup B = \{x \in X \mid x \in A \ \vee \ x \in B\} \ . \tag{1.47}$$

With the characteristic functions $\mu_A(x)$ and $\mu_B(x)$ of the sets A and B, the characteristic function $\mu_{A \cup B}(x)$ of the union $A \cup B$ can be defined as

$$\mu_{A \cup B}(x) = \begin{cases} 1 & \text{if} \quad \mu_A(x) = 1 \ \vee \ \mu_B(x) = 1 \\ 0 & \qquad \text{otherwise} \quad . \end{cases} \tag{1.48}$$

The generalization of (1.48) to the definition of the union $R \cup S$ of two n-ary relations R and S, $R, S \subseteq X_1 \times X_2 \times \ldots \times X_n$, yields

$$\mu_{R \cup S}(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if} \quad \mu_R(x_1, x_2, \ldots, x_n) = 1 \\ & \vee \quad \mu_S(x_1, x_2, \ldots, x_n) = 1 \\ 0 & \qquad \text{otherwise} \quad . \end{cases} \tag{1.49}$$

Difference

The *set difference* $B \setminus A$ of the sets A and B, $A, B \subseteq X$, is the set of all elements of B that are not members of A. Symbolically, we can write

$$B \setminus A = \{x \in X \mid x \in B \ \wedge \ x \notin A\} \ . \tag{1.50}$$