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Preface

Simply, quantum calculus is ordinary classical calculus without the notion of limits. It defines *q-calculus* and *h-calculus*. Here h ostensibly stands for Planck's constant, while q stands for quantum. A pioneer of q-calculus in approximation theory is the former Professor Alexandru Lupas [117], who first introduced the q-analogue of Bernstein polynomials. Ten years later Phillips [133] introduced another generalization on Bernstein polynomials [113] based on q-integers. Ostrovska [125, 127] studied q-Bernstein polynomials. After that several researchers have estimated the approximation properties of several operators. This book is an attempt to compile and present some papers on q-calculus in approximation theory.

We divide the book into seven chapters. In Chap. 1, we mention some notations and basic definitions of q-calculus, which will be used throughout the book. We also present the generating functions of some of the important q-basis functions. In Chap. 2, we present some discrete q-operators, which include the q-Bernstein polynomials, q-Baskakov operators, q-Szász operators, q-Blemian–Butzer–Hahn operators, and q-Meyer–König and Zeller operators. We present the approximation properties of such operators.

In Chap. 3, we present the q-analogue of integral operators which include q-Picard and q-Weierstrass-type singular integral operators and study their rate of convergence and weight approximation. We also discuss error estimation and global smoothness preservation property of such operators. In the last section of this chapter, we study generalized Picard operators and pointwise convergence, order of pointwise convergence, and norm convergence of the generalized operators. In the last section, we study the q-Meyer–König–Zeller–Durrmeyer operators and estimate the moments and some direct results.

In Chap. 4, we study the integral modifications of Bernstein operators using the q-beta functions of the first kind. We present the approximation properties of the q-Bernstein–Kantorovich operators, q-Bernstein–Durrmeyer polynomials, discretely defined q-Durrmeyer-type operators, and genuine q-Bernstein–Durrmeyer operators. We mention the moment estimation, direct results, and the limiting convergence of such operators. We have also included a section on fuzzy approximation and applications.

In Chap. 5, we discuss some other recently introduced q-integral operators on the positive real axis. To tackle such operators, we generally use q-beta functions of the second kind. This chapter includes q-Baskakov–Durrmeyer operators, q-Szász-beta operators, q-Szász–Durrmeyer operators, and q-Phillips operators. We present moments, recurrence relations for moments, asymptotic formula, and weighted approximations for such operators.

In Chap. 6, we study the statistical convergence of the q-operators. We mention results for a general class of positive linear operators and present statistical approximation properties in weighted space. We also present the results for q-Szász–King-type operators and q-Baskakov–Kantorovich operators and the study rate of convergence.

In the last chapter, we present the quantitative Voronovskaja-type estimate for certain q-Durrmeyer polynomials. In this way, we put in evidence the overconvergence phenomenon for these q-Durrmeyer polynomials, namely, the extensions of approximation properties (with quantitative estimates) from the real interval $[0,1]$ to compact disks in the complex plane. Also, we study the complex q-Gauss–Weierstrass integral operators. We show that these operators are an approximation process in some subclasses of analytic functions giving Jackson-type estimates in approximation. Furthermore, we give q-calculus analogues of some shape-preserving properties for these operators satisfied by the classical complex Gauss–Weierstrass integral operators.

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Introduction

Nowadays there is a significant increase of activities in the area of q-calculus due to its applications in various fields such as mathematics, mechanics, and physics. In 1910, Jackson [103] defined and studied the q-integral. He was the first to develop q-integral in a systematic way. Later the integral representations of q-gamma and q-beta functions were proposed by De Sole and Kac [49].

The applications of q-calculus in the area of approximation theory were initiated by Lupas [117], who first introduced q-Bernstein polynomials. Also in the last decade, Phillips [133] proposed other q-Bernstein polynomials, which became popular. Later several researchers obtained the interesting properties of q-Bernstein polynomials and their Durrmeyer variants; we mention some of the papers in this direction as [45, 86, 94, 129, 130]. The q-Bernstein–Durrmeyer-type operators are based on q-beta function of the first kind. The approximation of vector-valued functions by q-Durrmeyer operators with applications to random and fuzzy approximation was discussed by Gal and Gupta [77]. Another important operator, namely, q-Bleimann, Butzer, and Hahn operators, was discussed in [27] and also in [10, 60, 120]. The q-Baskakov operators in two different setups were proposed and studied in [30, 32]; also some direct results in terms of Ditzian–Totik modulus of continuity were discussed in [63]. Recently we [31] proposed the q-analogue of Baskakov–Durrmeyer operators, which is based on q-improper integral, namely, q-analogue of beta function of the second kind. Another important q-generalization of the well-known Szász–Mirakyan operators was proposed by Aral [25] and studied in detail by Aral and Gupta [29]. The authors have also proposed the q-analogue of Szász–Mirakyan–Durrmeyer operators in [33]. Several mixed q-analogues of hybrid summation-integral-type operators were proposed, some of them are discussed in this book.

Other q-analogues of integral operators are the q-Picard and q-Weierstrass-type singular integral operators. It can be observed that the q-Picard and the q-Gauss–Weierstrass singular integral operators give better approximation results than the classical ones. Trif [150] studied some approximation properties of the operators $\widehat{M}_{n,q}f(x)$. Also, Dogru and Gupta [55] proposed some other bivariate q-Meyer–König and Zeller operators having different test functions and established some approximation properties. Govil and Gupta [84] considered q-Meyer–König–Zeller–Durrmeyer operators which are discussed here.

In the recent years, many researchers have studied the statistical convergence for linear positive operators. Here, we present statistical convergence results for a general class of operators. We also discuss results for q-Szasz–King-type operators and q-Baskakov–Kantorovich operators and study rate of convergence.

For the quantitative Voronovskaja-type estimates for certain complex operators, we put in evidence the overconvergence phenomenon, namely, the extensions of approximation properties (with quantitative estimates) from the real interval to compact disks in the complex plane. We study complex operators of q-Durrmeyer-type and complex q-Gauss–Weierstrass integral operators. We show that these operators satisfy approximation process in some subclasses of analytic functions giving Jackson-type estimates in approximation. Furthermore, we give q-calculus analogues of some shape-preserving properties for these operators satisfied by the classical complex Gauss–Weierstrass integral operators.

Chapter 1
Introduction of q-Calculus

In the field of approximation theory, the applications of q-calculus are new area in last 25 years. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupas in the year 1987. In 1997 Phillips considered another q-analogue of the classical Bernstein polynomials. Later several other researchers have proposed the q-extension of the well-known exponential-type operators which includes Baskakov operators, Szász–Mirakyan operators, Meyer–König–Zeller operators, Bleiman, Butzer and Hahn operators (abbreviated as BBH), Picard operators, and Weierstrass operators. Also, the q-analogue of some standard integral operators of Kantorovich and Durrmeyer type was introduced, and their approximation properties were discussed. This chapter is introductory in nature; here we mention some important definitions and notations of q-calculus. We give outlines of q-integers, q-factorials, q-binomial coefficients, q-differentiations, q-integrals, q-beta and q-gamma functions. We also mention some important q-basis functions and their generating functions.

1.1 Notations and Definitions in q-Calculus

In this section we mention some basic definitions of q-calculus, which would be used throughout the book.

Definition 1.1. Given value of $q > 0$, we define the q-integer $[n]_q$ by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases},$$

for $n \in \mathbb{N}$.

We can give this definition for any real number λ. In this case we call $[\lambda]_q$ a q-real.

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Definition 1.2. Given the value of $q > 0$, we define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases},$$

for $n \in \mathbb{N}$.

Definition 1.3. We define the q-binomial coefficients by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}, \quad 0 \le k \le n, \tag{1.1}$$

for $n, k \in \mathbb{N}$.

The q-binomial coefficient satisfies the recurrence equations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{1.2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{1.3}$$

Definition 1.4. The q-analogue of $(1+x)_q^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\ldots\left(1+q^{n-1}x\right) & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined as

$$(x;q)_0 = 1, (x;q)_n = \prod_{i=0}^{n-1}(1-q^i x), (x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x).$$

Definition 1.5. The Gauss binomial formula:

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j x^{n-j}.$$

Definition 1.6. The Heine's binomial formula:

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n]_q[n+1]_q \ldots [n+j-1]_q}{[j]_q!} x^j.$$

Also, we have the following important property:

$$x^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (x-1)_q^j.$$

1.2 q-Derivative

Definition 1.7. The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0, \tag{1.4}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1} D_q f(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} = \frac{df(x)}{dx}$$

if f is differentiable.

It is obvious that the q-derivative of a function is a linear operator. That is, for any constants a and b, we have

$$D_q \{af(x) + bg(x)\} = aD_q \{f(x)\} + bD_q \{g(x)\}.$$

Now we calculate the q-derivative of a product at $x \neq 0$, using Definition 1.7, as

$$\begin{aligned} D_q \{f(x) g(x)\} &= \frac{f(qx) g(qx) - f(x) g(x)}{(q-1)x} \\ &= \frac{f(qx) g(qx) - f(qx) g(x) + f(qx) g(x) - f(x) g(x)}{(q-1)x} \\ &= \frac{f(qx)(g(qx) - g(x))}{(q-1)x} + \frac{(f(qx) - f(x)) g(x)}{(q-1)x} \\ &= f(qx) D_q g(x) + D_q f(x) g(x). \end{aligned}$$

We interchange f and g and obtain

$$D_q \{f(x) g(x)\} = f(x) D_q g(x) + D_q f(x) g(qx). \tag{1.5}$$

The Leibniz rule for the q-derivative operator is defined as

$$D_q^{(n)}(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{(k)} f(xq^{n-k}) D_q^{(n-k)} g(x).$$

If we apply Definition 1.7 to the quotient $f(x)$ and $g(x)$, we obtain

$$
\begin{aligned}
D_q\left\{\frac{f(x)}{g(x)}\right\} &= \frac{1}{(q-1)x}\left\{\frac{f(qx)}{g(qx)}-\frac{f(x)}{g(qx)}+\frac{f(x)}{g(qx)}-\frac{f(x)}{g(x)}\right\}\\
&= \frac{1}{g(qx)}\left\{\frac{f(qx)-f(x)}{(q-1)x}\right\}+\frac{1}{(q-1)x}\left\{\frac{f(x)g(x)-f(x)g(qx)}{g(qx)g(x)}\right\}\\
&= \frac{1}{g(qx)}D_qf(x)+\frac{f(x)}{g(qx)g(x)}\left\{\frac{g(x)-g(qx)}{(q-1)x}\right\}\\
&= \frac{g(x)D_qf(x)-f(x)D_qg(x)}{g(qx)g(x)}. \qquad (1.6)
\end{aligned}
$$

The above formula can also be written as

$$
D_q\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(qx)D_qf(x)-f(qx)D_qg(x)}{g(qx)g(x)}.
$$

Note that there does not exist a general chain rule for q-derivative. We can give a chain rule for function of the form $f(u(x))$, where $u=u(x)=\alpha x^{\beta}$ with α, β being constant. For this chain rule, we can write

$$
\begin{aligned}
D_q\{f(u(x))\} &= D_q\left\{f\left(\alpha x^{\beta}\right)\right\}\\
&= \frac{f\left(\alpha q^{\beta}x^{\beta}\right)-f\left(\alpha x^{\beta}\right)}{(q-1)x}\\
&= \frac{f\left(\alpha q^{\beta}x^{\beta}\right)-f\left(\alpha x^{\beta}\right)}{\alpha q^{\beta}x^{\beta}-\alpha x^{\beta}}.\frac{\alpha q^{\beta}x^{\beta}-\alpha x^{\beta}}{(q-1)x}\\
&= \frac{f\left(q^{\beta}u\right)-f(u)}{q^{\beta}u-u}.\frac{u(qx)-u(x)}{(q-1)x}
\end{aligned}
$$

and, hence,

$$
D_q\{f(u(x))\}=\left(D_{q^{\beta}}f\right)(u(x))D_q(u(x)).
$$

Proposition 1.1. *For $n\geq 1$,*

$$
D_q(1+x)_q^n=[n]_q(1+qx)_q^{n-1}
$$

$$
D_q\left\{\frac{1}{(1+x)_q^n}\right\}=-\frac{[n]_q}{(1+x)_q^{n+1}}.
$$

Proof. According to the definition of q-derivative we have

$$\begin{aligned} D_q(1+x)_q^n &= \frac{(1+qx)_q^n-(1+x)_q^n}{(q-1)x} \\ &= (1+qx)_q^{n-1}\frac{\{(1+q^nx-(1+x)\}}{(q-1)x} \\ &= [n]_q\,(1+qx)_q^{n-1}. \end{aligned}$$

According to (1.6), we have

$$\begin{aligned} D_q\left\{\frac{1}{(1+x)_q^n}\right\} &= -\frac{D_q(1+x)_q^n}{(1+qx)_q^n(1+x)_q^n} \\ &= -\frac{[n]_q}{(1+q^nx)\,(1+x)_q^n} \\ &= -\frac{[n]_q}{(1+x)_q^{n+1}}. \end{aligned}$$

■

Remark 1.1. Suppose $n \geq 1$ and $a,b,r,s \in \Re$, then by simple computation, we immediately have the following:

$$D_q(a+bx)_q^n = [n]_q b(a+bqx)_q^{n-1},$$

$$D_q(ax+b)_q^n = [n]_q a(ax+b)_q^{n-1},$$

and

$$D_q\frac{(1+ax)_q^r}{(1+bx)_q^s} = [r]_q a\frac{(1+aqx)_q^{r-1}}{(1+bqx)_q^s} - b[s]_q\frac{(1+ax)_q^r}{(1+bx)_q^{s+1}}.$$

1.3 q-Series Expansions

Theorem 1.1. *For* $|x|<1$, $|q|<1$,

$$\sum_{k=0}^{\infty}\frac{(1-a)_q^k}{(1-q)_q^k}x^k = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty},$$

where $(1-x)_q^\infty = \prod_{k=0}^{\infty}\left(1-q^kx\right)$.

Proof. Let

$$f_a(x) = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k.$$

Clearly

$$\begin{aligned}\frac{f_a(x) - f_a(qx)}{x} &= \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} \left(1-q^k\right) x^{k-1} \\ &= (1-a) \sum_{k=1}^{\infty} \frac{(1-aq)_q^{k-1}}{(1-q)_q^{k-1}} x^{k-1} \\ &= (1-a) \sum_{k=0}^{\infty} \frac{(1-aq)_q^k}{(1-q)_q^k} x^k = (1-a) f_a(qx)\end{aligned}$$

or

$$f_a(x) - f_a(qx) = (1-a) x f_a(qx).$$

Also

$$\begin{aligned}f_a(x) - f_a(qx) &= \sum_{k=0}^{\infty} \frac{(1-qa)_q^{k-1}}{(1-q)_q^k} \left(1-a-1+aq^k\right) x^k \\ &= -ax f_{aq}(x)\end{aligned}$$

or

$$f_a(x) = (1-ax) f_{aq}(x).$$

Combining the above two equations, we get

$$f_a(x) = \frac{1-ax}{1-x} f_a(qx).$$

Iterating this relation n times and letting $n \to \infty$ we have

$$f_a(x) = \frac{(1-ax)_q^n}{(1-x)_q^n} f_a(q^n x) = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}.$$

Thus we have the desired result. ∎

Corollary 1.1. *(a) Taking $a = 0$ in Theorem 1.1, we have*

$$\sum_{k=0}^{\infty} \frac{x^k}{(1-q)_q^k} = \frac{1}{(1-x)_q^\infty}, \qquad |x| < 1, |q| < 1.$$

(b) Replacing a with $\frac{1}{a}$, and x with ax, and then taking $a = 0$ in Theorem 1.1, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^n q^{\frac{k(k-1)}{2}} x^k}{(1-q)_q^k} = (1-x)_q^{\infty}, \qquad |q| < 1.$$

(c) Taking $a = q^N$ in Theorem 1.1, we have

$$\sum_{k=0}^{\infty} \begin{bmatrix} N-k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(1-x)_q^N}, \qquad |x| < 1.$$

We consider Corollary 1.1(a). We can write

$$\sum_{k=0}^{\infty} \frac{x^k}{(1-q)_q^k} = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right)}$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{[k]_q!}$$

which resembles Taylor's expansion of classical exponential function e^x.

Definition 1.8. A q-analogue of classical exponential function e^x is

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}.$$

Using Corollary 1.1, (a) and (b), we see that

$$e_q\left(\frac{x}{1-q}\right) = \frac{1}{(1-x)_q^{\infty}}$$

and

$$e_q(x) = \frac{1}{(1-(1-q)x)_q^{\infty}}. \tag{1.7}$$

Definition 1.9. Another q-analogue of classical exponential function is

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}. \tag{1.8}$$

The q-exponential functions satisfy following properties:

Lemma 1.1. *(a)* $D_q e_q(x) = e_q(x), \qquad D_q E_q(x) = E_q(qx).$
(b) $e_q(x) E_q(-x) = E_q(x) e_q(-x) = 1.$